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World Maps on the August Epicycloidal Conformal Projection

ERWIN SCHMID

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ABSTRACT. Mapping equations and their inverses are developed in simplified form for the August two-cusped epicycloidal conformal projection and for the Lagrange family of conformal projections bounded by circular arcs.

1. INTRODUCTION

F. August (1873) developed the theory of a conformal projection of the whole sphere within a two-cusped epicycloid. The National Geodetic Survey (N.O.S.–N.O.A.A.) was recently requested by the Smithsonian Institution to compute and plot various skewed aspects of a map of the world on this projection; the following pages are a result of this effort. Aside from the language barrier and the limited accessibility of the cited reference there are other valid justifications for another and, in some respects, supplementary presentation of the subject. Probably the principal of these reasons is to shift the emphasis from the geometrical construction of the grid, which in those days was of primary interest to the cartographer, to the explicit formulation of the analytical mapping equations which today's computer-programmer needs to produce a map of not only the grid but also of the outlines of continents, boundaries of countries, etc. Although August's presentation is complete and rigorous, a cartographer not familiar with Complex Function Theory would have some difficulty in deriving the necessary programmable mapping equations with (ϕ, λ), i.e., latitude and longitude input from the cited reference. As a matter of fact, it is possible to develop these formulas by applying analytical geometry to August's geometrical construction presented in section 10.

August (1873) interchanges the X and Y axes of the complex plane, presumably to conform with the standard practice in German cartography. This practice has certain advantages in that azimuth, which is reckoned clockwise, can be identified with inclination in the formulas of analytical geometry and trigonometry. However, this convention has not been generally adopted in the United States; here we retain the standard practice in both function theory and cartography of designating the X-axis as the axis of abscissas. The distinction is not entirely trivial because occasional changes in sign are produced in going from one system to the other.

August begins his demonstration by developing the mapping equations (as they are called in the theory of functions of a complex variable) of the meridional aspect of the stereographic projection of the unit sphere onto the complex plane. For cartographic purposes this complex plane can equally well be interpreted as the plane sheet in which the map is drawn. He then gives, without proof, the transformation on the coordinates of the stereographic projection that produce the ordinary Lagrange conformal projection of the sphere within a unit circle. In this projection the circumference of the circle represents both halves of the meridian of longitude 180° with respect to the central meridian, which is a diameter of the circle. A two-cusped epicycloid is generated by rolling a circle with radius = 1/2 on the circumference of the unit circle. By the methods of analytical geometry the equation of the epicycloid is then found in terms of the x, y coordinates of the circle, thus mapping the 180° meridian onto the epicycloid (fig. 1). According to the principle of analytical continuation these same equations therefore map the interior of the circle into the interior of the epicycloid conformally, with the exception only of the singular points of the transformation.

2. THE STEREOGRAPHIC PROJECTION

This well-known projection is the only truly perspective view of the sphere that is also conformal, and is of fundamental importance in complex function theory. The development of the corresponding mapping equations onto the plane of the complex variable $z = x + iy$ can be found in any textbook on the subject. For the meridional aspect, i.e., with the perspective center on the equator and the plane of projection parallel to the polar axis, these equations are:

$$\begin{cases} x = \dfrac{\sin\lambda\cos\phi}{1+\cos\lambda\cos\phi} \\ y = \dfrac{\sin\phi}{1+\cos\lambda\cos\phi}, \end{cases} \tag{1}$$

where x, y are the rectangular components of the complex number $z = x + iy$ and ϕ, λ are latitude and longitude on the sphere. The equations are scaled arbitrarily so that the points within the unit circle represent a unit hemisphere centered about a point of the equator, the origin of the x, y plane, with spherical coordinates $\phi = \lambda = 0$. The points on the other half of the sphere map into the exterior of the unit circle.

To solve these equations for ϕ and λ, i.e., to obtain the inverse of the transformation (1), divide the first equation by the second to obtain

$$\frac{x}{y} = \frac{\sin\lambda}{\tan\phi} \tag{2}$$

or $\sin\lambda = \dfrac{x\sin\phi}{y\cos\phi}$. Substituting $\cos\lambda = \sqrt{1-\sin^2\lambda}$ $= \dfrac{\sqrt{y^2\cos^2\phi - x^2\sin^2\phi}}{y\cos\phi}$ in the second eq (1) results in

$$1 = \frac{\sin\phi}{y + \sqrt{y^2\cos^2\phi - x^2\sin^2\phi}}.$$

Rationalizing this equation and substituting $(y^2 - y^2\sin^2\phi)$ for $y^2\cos^2\phi$ results ultimately in $\sin\phi = \dfrac{2y}{1+x^2+y^2}$.

Similarly, by using $\tan\phi = \dfrac{y\sin\lambda}{x}$ from (2), ϕ can be eliminated from either of (1). The combined result, eq (3), is then the required inverse of the meridional stereographic eq (1).

$$\begin{cases} \sin\phi = \dfrac{2y}{1+x^2+y^2} \\ \tan\lambda = \dfrac{2x}{1-x^2-y^2}. \end{cases} \tag{3}$$

In addition to enabling one to find the spherical coordinates of a point from given x, y coordinates on the map, (3) are also the equations in the mapping plane of the parallels and meridians of the sphere, respectively. From the form of the equations, it is apparent that both sets of curves are circles, with centers on the coordinate axes. It is a characteristic property of this projection that all circles on the sphere map into circles in the plane.

3. THE CAUCHY-RIEMANN EQUATIONS

In complex function theory the criterion for the conformality of a mapping is basic for the definition of analytical functions and is, from a formalistic standpoint at any rate, considerably less complicated than the corresponding concept in differential geometry. In the text books on complex variables it is shown that an analytic function

$$f(z) = f(x+iy) = u(x,y) + iv(x,y) \tag{4}$$

of the complex variable z must satisfy the Cauchy-Riemann equations

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}, \end{cases} \tag{5}$$

and conversely.

Translated into cartographic terms, this means that if a conformal projection such as the stereographic is given in terms of Cartesian coordinates x, y, then a projection with coordinates X, Y, where the latter are each functions of x and y, will be conformal if, and only if,

$$\begin{cases} \dfrac{\partial X}{\partial x} = \dfrac{\partial Y}{\partial y} \\ \dfrac{\partial X}{\partial y} = -\dfrac{\partial Y}{\partial x}. \end{cases} \tag{6}$$

This test is not immediately applicable to the stereographic projection as given in (1) since the x and y coordinates are given there as functions of ϕ and λ which are not, as such, Cartesian coordinates of a conformal projection of the sphere. The modification necessary to make the conditions (6) applicable to a projection defined in terms of ϕ and λ is shown in section 6.

4. THE LAGRANGE PROJECTIONS

Lagrange, as cited by Scheffers (1902), set himself the problem of finding all conformal projections in which the meridians and parallels are arcs of circles. For the meridional aspect, it seems a reasonable presumption that the mutually orthogonal family of circles used to represent parallels and meridians arising from the definition of the stereographic projection, i.e., the circles (3) will again serve as the grid but with different designations. Scheffers shows that this family is indeed a necessary consequence of the statement of the problem. Consequently the Lagrangian projections can be thought of as a generalization of the stereographic meridional projection with the meridans λ of this latter projection relabeled as $n\lambda$ where n is any positive number, including fractions. For $n=2$, for instance, the two halves of the circumference of the unit circle become the 180° meridians E and W respectively, so that the whole sphere is mapped within the circle. This particular case is usually designated as the Lagrange Projection and, as such, credited to J. H. Lambert by some writers. For other values of n it is apparent that any portion of the sphere symmetrical to a central meridian or the whole surface of the sphere will be mapped between two specified arcs of circles. There is no loss of generality in specifying that the circles intersect on the y axis at unit distance from the origin, these two points being the poles in the projection.

Having specified that we want to make the substitution $\lambda^* = \frac{\lambda}{n}$ for λ in equations (1), the only remaining condition to be satisfied is to determine a corresponding transformation ϕ^* on ϕ to produce a conformal projection. To put it in geometrical terms, the parallel circles of the stereographic projection must remain parallels but their spacing will be different, the orthogonality of the two families of circles being necessary but not sufficient for conformality. We shall incidentally prove the latitude transformation $\sin \phi^* = \tan \frac{\phi}{2}$ for the case $n = 2$, which is usually given in the textbooks without proof.

In analytical terms, our problem is: Given eq (1), in which $x = x(\lambda, \phi)$, $y = y(\lambda, \phi)$, then the family of Lagrange projections with parameter n is

$$\begin{cases} X = x(\lambda^*, \phi^*) \\ Y = y(\lambda^*, \phi^*), \text{ where } \lambda^* = \dfrac{\lambda}{n}. \end{cases} \tag{7}$$

Is there a function ϕ^* of ϕ alone such that eq (7) satisfies the Cauchy Riemann conditions (6)?

We must have
$$\begin{cases} \dfrac{\partial X}{\partial \lambda^*} = \dfrac{\partial Y}{\partial \phi^*} \\ \dfrac{\partial X}{\partial \phi^*} = -\dfrac{\partial Y}{\partial \lambda^*}. \end{cases} \tag{8}$$

In the first equation, $\dfrac{\partial X}{\partial \lambda^*} = \dfrac{\partial x}{\partial \lambda} \cdot \dfrac{\partial \lambda}{\partial \lambda^*}$; here $\dfrac{\partial x}{\partial \lambda}$ is obtained by differentiating the first of eq (1) with respect to λ. The result is

$$\frac{\partial x}{\partial \lambda} = \frac{\cos \phi\ (\cos \lambda + \cos \phi)}{(1 + \cos \lambda \cos \phi)^2}.$$

From the relation $\lambda = n\lambda^*$ used in (7) we have $\dfrac{\partial \lambda}{\partial \lambda^*} = n$. Hence

$$\frac{\partial X}{\partial \lambda^*} = \frac{n \cos \phi\ (\cos \lambda + \cos \phi)}{(1 + \cos \lambda \cos \phi)^2}. \tag{9}$$

Similarly $\dfrac{\partial Y}{\partial \phi^*} = \dfrac{\partial y}{\partial \phi} \cdot \dfrac{\partial \phi}{\partial \phi^*}$. With $\dfrac{\partial y}{\partial \phi}$ obtained by differentiating the second of eq (1),

$$\frac{\partial Y}{\partial \phi^*} = \frac{\cos \phi + \cos \lambda}{(1 + \cos \lambda \cos \phi)^2} \frac{\partial \phi}{\partial \phi^*}. \tag{10}$$

Equating (9) and (10) we have the necessary condition to be satisfied for conformality, i.e.,

$$\frac{\partial \phi}{\partial \phi^*} = n \cos \phi = \frac{d\phi}{d\phi^*}, \tag{11}$$

where the partial derivative is equal to the total derivative because ϕ^* was postulated as independent of λ. The condition for conformality is therefore the ordinary differential equation

$$d\,\phi^* = \frac{1}{n} \sec \phi\ d\,\phi$$

with the solution

$$\phi^* = \frac{1}{n} \int \sec \phi \ d\ \phi = \frac{1}{n} \ln\ (\sec \phi + \tan \phi) \tag{12}$$

which can be verified by differentiation.

Proceeding in similar fashion with the second of the condition (8) it is seen to be satisfied also by the relation (11). The relation is therefore sufficient as well as necessary, and we have proved that the substitution of $\frac{1}{n} \ln\ (\sec \phi + \tan \phi)$ for ϕ and $\frac{\lambda}{n}$ for λ in the stereographic formula (1) produces the conformal General Lagrange Projection with parameter n. We shall write (12) as

$$\phi^* = \frac{\mu}{n} \tag{13}$$

where $\mu = \ln(\sec\phi + \tan\phi)$ is designated in cartography as isometric latitude because, as demonstrated in texts on differential geometry, μ and λ are isometric or isothermal parameters on the sphere. The latitude function μ = const. and λ = const. plotted in an x, y plane as lines y = const. and x = const., respectively, produce the conformal Mercator projection. Alternative expressions for μ sometimes used for computational or analytical purposes are:

$$\begin{cases} \mu = \ln \tan\left(\dfrac{\pi}{4} + \dfrac{\phi}{2}\right) \\ \mu = \dfrac{1}{2} \ln \dfrac{1 + \sin\phi}{1 - \sin\phi}. \end{cases} \tag{14}$$

These can be derived from $\mu = \ln(\sec\phi + \tan\phi)$ by straightforward trigonometric formulas.

The Cauchy-Riemann equations are the basis for the theorem in complex variable theory that every conformal mapping in the complex plane is an analytic function of any other conformal mapping. Which mapping is considered the basic one is a matter of choice. Because of its great antiquity and simple geometric construction, this distinction is commonly accorded the stereographic projection. However, the fact that the geographic grid of the sphere coincides with the x, y lines of the mapping plane in the Mercator projection makes it seem likely that, on the whole, the totality of conformal projections can be expressed most conveniently and compactly in terms of the complex variable $z = \lambda + i\mu$.

From the definition of n, it is evident that for $n = 1$ eq (7) represent the stereographic projection, since $\lambda^* = \frac{\lambda}{n} = \lambda$. From (12) and (13), $\phi^* = \frac{\mu}{n} = \mu$. Hence, eq (7) express the stereographic X and Y coordinates in terms of λ and μ

$$\begin{cases} X = x(\lambda, \mu) \\ Y = y(\lambda, \mu) \end{cases} \tag{15}$$

To deduce the form of the function on the right of eq (15) from the left, which we will assume expressed in terms of ϕ and λ as in (1), we note first that no change is necessary in the functions of λ involved. To transform the trigonometric functions of ϕ in (1) into functions of μ, we use the Gudermannian transformations (Dwight 1934):

$$\begin{aligned} \sin\phi &= \tanh\mu \\ \cos\phi &= 1/\cosh\mu. \end{aligned} \tag{16}$$

The result of these substitutions into (1) gives the desired form of the stereographic as a function of λ and μ

$$\begin{cases} X = \dfrac{\sin\lambda}{\cosh\mu + \cos\lambda} \\ Y = \dfrac{\sinh\mu}{\cosh\mu + \cos\lambda}. \end{cases} \tag{17}$$

From (7) and (13), it is also evident that to pass from the stereographic to the general Lagrange projection all that is necessary is to divide both λ and μ in (17) by n. Thus the Lagrange projection is

$$\begin{cases} X = \dfrac{\sin\dfrac{\lambda}{n}}{\cosh\dfrac{\mu}{n} + \cos\dfrac{\lambda}{n}} \\ Y = \dfrac{\sinh\dfrac{\mu}{n}}{\cosh\dfrac{\mu}{n} + \cos\dfrac{\lambda}{n}} \end{cases} \tag{18}$$

for all values of n.

5. INVERSE OF THE GENERAL LAGRANGE PROJECTION

From (18), by division

$$\frac{X}{Y} = \frac{\sin\dfrac{\lambda}{n}}{\sinh\dfrac{\mu}{n}} \tag{19}$$

Hence $\sin\dfrac{\lambda}{n} = \dfrac{X\sinh\dfrac{\mu}{n}}{Y}$ and $\cos\dfrac{\lambda}{n}$

$$= \left(1 - \sin^2\frac{\lambda}{n}\right)^{1/2} = \frac{\left(Y^2 - X^2\sinh^2\dfrac{\mu}{n}\right)^{1/2}}{Y}$$

Substituting these quantities into the second of eq (18) and proceeding in a manner similar to that used to develop (3) results in

$$Y^2 - X^2\sinh^2\frac{\mu}{n} = \sinh^2\frac{\mu}{n} + Y^2\cosh^2\frac{\mu}{n} - 2Y\sinh\frac{\mu}{n}\cosh\frac{\mu}{n}.$$

For $Y^2 \cosh^2 \frac{\mu}{n}$ write $Y^2 + Y^2 \sinh^2 \frac{\mu}{n}$ and we have

$$(X^2 + Y^2 + 1) \sinh^2 \frac{\mu}{n} = 2Y \sinh \frac{\mu}{n} \cosh \frac{\mu}{n}$$

or

$$\tanh \frac{\mu}{n} = \frac{2Y}{X^2 + Y^2 + 1}$$

Similarly, by solving (19) for $\sinh \frac{\mu}{n}$ and substituting for $\cosh \frac{\mu}{n}$ in the first of eq (18), an expression for $\tan \frac{\lambda}{n}$ is obtained. The combined results are the required inverse of eq (18)

$$\left\{\begin{aligned} \tan \frac{\lambda}{n} &= \frac{2X}{1 - X^2 - Y^2} \\ \tanh \frac{\mu}{n} &= \frac{2Y}{1 + X^2 + Y^2}. \end{aligned}\right. \tag{20}$$

It is again apparent that for constant λ or μ, the inverse eq (20) are the equations of the meridians and parallels, respectively, and that they are all circles with centers on the coordinate axes as was specified in the statement of the problem.

In particular, for the stereographic projection ($n = 1$) the eq (20) are, as before in eq (3),

$$\left\{\begin{aligned} \tan \lambda &= \frac{2X}{1 - X^2 - Y^2} \\ \tanh \mu = \sin \phi &= \frac{2Y}{1 + X^2 + Y^2} \end{aligned}\right. \tag{21}$$

and for the common Lagrange projection ($n = 2$)

$$\left\{\begin{aligned} \tan \frac{\lambda}{2} &= \frac{2X}{1 - X^2 - Y^2} \\ \tanh \frac{\mu}{2} = \tan \frac{\phi}{2} &= \frac{2Y}{1 + X^2 + Y^2}. \end{aligned}\right. \tag{22}$$

The relation $\sin \phi = \tanh \mu$ in (21) is from (16) and can be demonstrated as follows. From the second of eq (14), it follows that

$$e^{2\mu} = \frac{1 + \sin \phi}{1 - \sin \phi}$$

and solving this for $\sin \phi$ we have

$$\sin \phi = \frac{e^{2\mu} - 1}{e^{2\mu} + 1}.$$

The right side of this equation equals $\tanh \mu$ by definition of the hyperbolic tangent.

From the first of eq (14)

$$e^{\mu} = \tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right) = \frac{1 + \tan \frac{\phi}{2}}{1 - \tan \frac{\phi}{2}}.$$

Solving this for $\tan \frac{\phi}{2}$ leads to the relation $\tanh \frac{\mu}{2} = \tan \frac{\phi}{2}$ in (22). Comparison of (21) and (22) indicates the effect of reducing μ to $\frac{\mu}{2}$ in going from the stereographic to the ordinary Lagrange projection directly in terms of the latitude ϕ. It is the substitution of $\tan \frac{\phi}{2}$ for $\sin \phi$ which August tacitly assumes in both his geometrical construction and in his analytical treatment of the ordinary Lagrange projection. For the general Lagrange projection (18), no such simple relation in terms of elementary functions of ϕ is apparent. However, in terms of μ, the transformation is quite general and formalistically simple.

6. THE CAUCHY-RIEMANN EQUATIONS IN TERMS OF ϕ, λ

We have used the theorem from complex variable analysis that, if mapping equations $x(\lambda, \mu)$ and $y(\lambda, \mu)$ are expressed in terms of the isometric variables λ and μ and satisfy the Cauchy-Riemann equations

$$\left\{\begin{aligned} \frac{\partial x}{\partial \lambda} &= \frac{\partial y}{\partial \mu} \\ \frac{\partial x}{\partial \mu} &= -\frac{\partial y}{\partial \lambda}, \end{aligned}\right. \tag{23}$$

then the mapping transformation will be conformal, and conversely.

Since $\frac{\partial x}{\partial \mu} = \frac{\partial x}{\partial \phi}\frac{\partial \phi}{\partial \mu}$, $\frac{\partial y}{\partial \mu} = \frac{\partial y}{\partial \phi}\frac{\partial \phi}{\partial \mu}$ and, from (11) and (13) $\frac{\partial \mu}{\partial \phi} = \sec \phi$, we have by substitution of these quantities in (23)

$$\begin{aligned} \frac{\partial x}{\partial \lambda} &= \frac{\partial y}{\partial \phi} \cos \phi \\ \frac{\partial x}{\partial \phi} \cos \phi &= -\frac{\partial y}{\partial \lambda} \end{aligned} \tag{24}$$

as a set of Cauchy-Riemann conditions directly in terms of ϕ and λ. In cartography, mapping equations are generally expressed as functions of ϕ and λ, and the form (24) is therefore directly applicable as a test for conformality of the transformation or, as the case may be, the condition to be satisfied to make the mapping conformal. For example, differentiating (1) with respect to ϕ and λ, the stereographic projection satisfies both conditions (24) and is therefore conformal. On the other hand, for the Sanson Sinusoidal projection

$$\begin{cases} x=\lambda \cos \phi \\ y=\phi \end{cases}$$

the first of conditions (24) is satisfied but not the second, so this map is not conformal; it is in fact area-equivalent.

7. PROJECTION OF THE SPHERE WITHIN A TWO-CUSPED EPICYCLOID

The epicycloid in figure 1 is a curve traced by a point U of a circle with radius 1/2 rolling on the unit circle centered at the origin of the x, y coordinate system. To make the north poles of the two projections coincide, August specifies that the point of the smaller circle that traces the curve shall be the point of contact of the two circles when the center of this smaller circle lies on the positive y axis. With no slippage present the locus U of the specified point of the rolling circle will be such that the arc NP on the stationary unit circle equals the arc PU on the smaller circle. Consequently $\angle$ PCU$=$ 2($\angle$ NOP) and $\angle$ UCD$=$2($\angle$ POX). Projecting OU on the coordinate axes, the coordinates (x_u, y_u) of U are

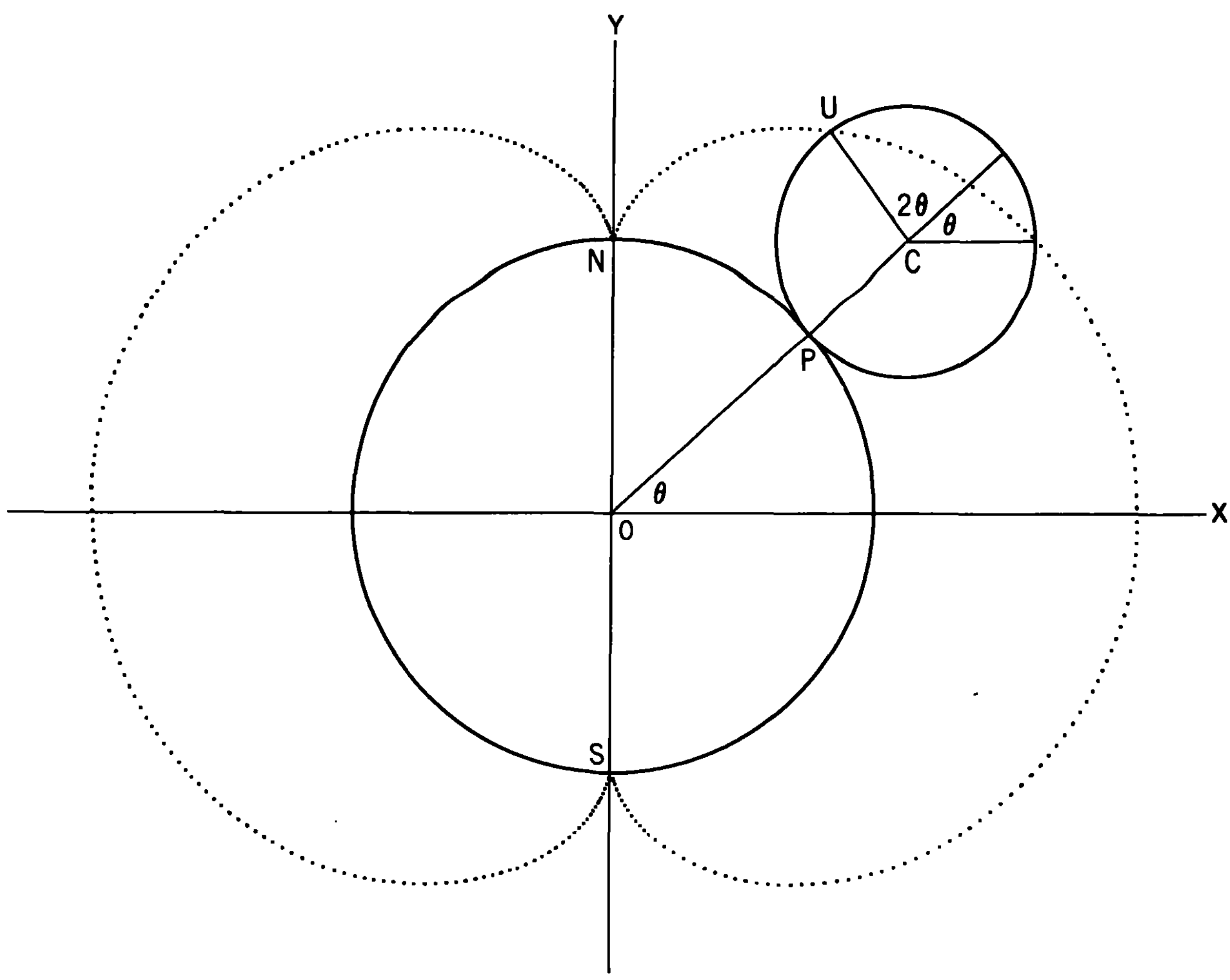

FIGURE 1.—Two-cusped epicycloid generated by a fixed point U on the circumference of a circle rolling on the exterior of the unit circle.

$$\begin{cases} x_u = \frac{3}{2}\cos\Theta + \frac{1}{2}\cos 3\Theta \\ y_u = \frac{3}{2}\sin\Theta + \frac{1}{2}\sin 3\Theta. \end{cases} \tag{25}$$

If we now interpret the plane of figure 1 as the complex plane, then the complex value u of U is $x_u + iy_u$ or

$$u = \frac{3}{2}\ (\cos\Theta + i\sin\Theta) + \frac{1}{2}\ (\cos 3\Theta + i\sin 3\Theta)$$

or, with DeMoivre's theorem,

$$u = \frac{1}{2}\ (3e^{i\Theta} + e^{3i\Theta}). \tag{26}$$

Furthermore, the point P on the unit circle has the complex value $p = e^{i\Theta}$ so that (26) becomes

$$u = \frac{1}{2}\ (3p + p^3) \tag{27}$$

which establishes a functional relation or mapping between the points of the unit circle and of the epicycloid. If we now interpret all points of the unit circle, both on the circumference and in the interior, as points of the ordinary Lagrange projection then, by the principle of analytic continuation, the relation (27) maps all points of the sphere (except singular points) from the unit circle into the region of the plane bounded by the epicycloid. Since, as we have shown, the Lagrange projection is conformal and, by (27), u is an analytic function of p (i.e., the transformation (27) satisfies the Cauchy-Riemann conditions), the August projection is therefore also conformal.

Designating the abscissas and ordinates of the Lagrangian points p as x and y respectively and of the corresponding August points u as X and Y, eq (27) will read

$$X + iY = \frac{1}{2}\ [3(x + iy) + (x + iy)^3]$$

or, by using the binomial theorem

$$X + iY = \frac{x}{2}\ (3 + x^2 - 3y^2) + i\,\frac{y}{2}\ (3 + 3x^2 - y^2). \tag{28}$$

Since the complex number on the left of (28) is equal to that on the right, we must have

$$\begin{cases} X = \frac{x}{2}\ (3 + x^2 - 3y^2) \\ Y = \frac{y}{2}\ (3 + 3x^2 - y^2) \end{cases} \tag{29}$$

which are the equations for the August two-cusped epicycloidal projection, where the auxiliary variables x, y are the coordinates of the ordinary Lagrange projection computed from (18) with $n = 2$, or from (1) by replacing λ with $\frac{\lambda}{2}$, $\sin\phi$ with $\tan\frac{\phi}{2}$, and $\cos\phi$ with $\left(1 - \tan^2\frac{\phi}{2}\right)^{1/2}$

As August points out, if the function μ in the formulas is replaced by the corresponding isometric parameter $\mu(e)$ on the ellipsoid, i.e., if we substitute for μ the quantity

$$\mu(e) = \ln\left[\left(\frac{1 - e\sin\phi}{1 + e\sin\phi}\right)^{e/2} \tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right)\right]$$

where e is the eccentricity of the meridian ellipse, the ellipsoid will be mapped with the same formulas. However, further modifications are necessary for skewed (oblique) aspects of a projection because on an ellipsoid of revolution, unlike the sphere, the geographic poles occupy a unique position on the surface.

8. INVERSION OF THE EPICYCLOIDAL COORDINATES

The problem of finding equivalent geographic coordinates (ϕ, λ) on the sphere from given epicycloidal coordinates X and Y as computed from (29), requires more sophisticated formulas from complex variable theory than have been used in the preceding sections. To make the argument easier for students to follow, we adopt a more conventional notation in this section. Thus, in (27) u designates the complex variable defining the epicycloidal point and p the corresponding Lagrange point. We shall now use the equivalent relation

$$w = \frac{1}{2}\ (3z + z^3) \qquad \text{with } w = u + iv, \quad z = x + iy \tag{30}$$

where u, v are the X, Y of the previous section and x, y are unchanged. The transformation (30) is a mapping of the z-plane onto the w-plane.

The inverse relation, mapping the w-plane back to the z-plane can be effected by reversing the series, i.e., by expressing z as an infinite series in powers of the complex variable w, or by iterative methods. The resulting processes, however, are cumbersome and imprecise because of slow convergence in some regions of the configuration. A rigorous solution for z in terms of w in closed form can be found by the following device:

The solution of the cubic equation

$$z^3 + 3z - 2w = 0,$$

which is the functional relation (30), is, by use of Cardan's formula for complex variables

$$z=p+q \qquad \text{with } p=(w+\sqrt{w^2+1})^{1/3} \quad q=(w-\sqrt{w^2+1})^{1/3} \tag{31}$$

(See p. 339, Townsend 1930).

The equation has two further roots, but these represent branches of the function z which are not pertinent to the present problem. Let

$$w=\sinh 3\tau, \quad \text{where } 3\tau=3\xi+3i\eta. \tag{32}$$

Then

$$\sinh 3\tau=\sinh 3\xi \cos 3\eta + i\cosh 3\xi \sin 3\eta \tag{33}$$

so that, with the relation $\cosh^2\tau = 1+\sinh^2\tau$,

$$\begin{cases} p=(\sinh 3\tau+\cosh 3\tau)^{1/3}=(e^{3\tau})^{1/3}=e^{\tau} \\ q=(\sinh 3\tau-\cosh 3\tau)^{1/3}=(-e^{-3\tau})^{1/3}=-e^{-\tau} \end{cases}$$

$$\text{with } \tau=\xi+i\eta.$$

From (31), therefore,

$$z=p+q=e^{\tau}-e^{-\tau}=2\sinh\tau$$

$$=2(\sinh\xi\cos\eta+i\cosh\xi\sin\eta) \tag{34}$$

The various functional relations for hyperbolic and exponential functions can all be found, for example, in Dwight (1934). To find ξ and η we have from (32) and (33) the simultaneous equations

$$\begin{cases} u=\sinh 3\xi \cos 3\eta \\ v=\cosh 3\xi \sin 3\eta \end{cases} \tag{35}$$

Squaring these

$$u^2=\sinh^2 3\xi \cos^2 3\eta$$

$$v^2=\cosh^2 3\xi \sin^2 3\eta=(1+\sinh^2 3\xi)\sin^2 3\eta$$

$$=\left(1+\frac{u^2}{1-\sin^2 3\eta}\right)\sin^2 3\eta$$

Hence

$$\sin^4 3\eta-(1+v^2+u^2)\sin^2 3\eta+v^2=0,$$

a quadratic equation in $\sin^2 3\eta$ the solution for which is

$$\sin^2 3\eta=\frac{1}{2}\left(1+v^2+u^2-\sqrt{(1+v^2+u^2)^2-4v^2}\right) \tag{36}$$

where the minus sign before the radical is chosen to keep $\sin^2 3\eta$ in the range $0\leqslant \sin^2 3\eta \leqslant 1$. Since there is complete symmetry in the four quadrants of the map, no generality is lost by assuming u, v, and hence, ξ, η, positive. Extract the positive square root for $\sin 3\eta$, from which follows

$$\cosh 3\xi=\frac{v}{\sin 3\eta} \tag{37}$$

by eq (35).

From $\sin 3\eta$ and $\cosh 3\xi$ obtain 3η, 3ξ and, hence, η and ξ. From (34) the Lagrange coordinates corresponding to the given epicycloidal coordinates u, v are then

$$\begin{cases} X=2\sinh\xi\cos\eta \\ Y=2\cosh\xi\sin\eta \end{cases} \tag{38}$$

which, with (22), give the solution for ϕ and λ

$$\begin{cases} \tan\dfrac{\lambda}{2}=\dfrac{2X}{1-X^2-Y^2} \\ \tan\dfrac{\phi}{2}=\dfrac{2Y}{1+X^2+Y^2} \end{cases} \tag{22}$$

Since, from (38)

$$X^2+Y^2=4(\sinh^2\xi\cos^2\eta+\cosh^2\xi\sin^2\eta)$$

$$=4(\cosh^2\xi-\cos^2\eta)$$

the solution (22) can be written directly in terms of ξ and η as

$$\begin{cases} \tan\dfrac{\lambda}{2}=\dfrac{\sinh\xi\cos\eta}{1/4-(\cosh^2\xi-\cos^2\eta)} \\ \tan\dfrac{\phi}{2}=\dfrac{\cosh\xi\sin\eta}{1/4+(\cosh^2\xi-\cos^2\eta)} \end{cases} \tag{39}$$

9. NUMERICAL EXAMPLES

Use of the formulas presented here will be illustrated by some numerical examples computed on a 10-digit electronic desk calculator capable of computing trigonometric, hyperbolic, and exponential functions and the corresponding inverse functions. The numbers in parentheses refer to the pertinent equations in the text.

Ex. 1. The stereographic projection.
Spherical coordinates

Latitute $\phi = +50°$ (north)
Longitude $\lambda = +100°$ (east of central meridian)

$$\sin\phi = 0.76604\ 44431$$
$$\cos\phi = 0.64278\ 76097$$
$$\sin\lambda = 0.98480\ 77530$$
$$\cos\lambda = 0.17364\ 81777$$

(1)
$$\begin{cases} x = \dfrac{(0.98480\ 77530)\ (0.64278\ 76097)}{1+(-0.17364\ 81777)\ (0.64278\ 76097)} = 0.71255\ 70540 \\[2ex] y = \dfrac{.76604\ 44431}{1+(-0.17364\ 81777)\ (0.64278\ 76097)} = 0.86229\ 25911. \end{cases}$$

For the inverse

(3)
$$\begin{cases} \sin\phi = \dfrac{2(0.86229\ 25911)}{1+(0.71255\ 70540)^2+(0.86229\ 25911)^2} = 0.76604\ 44431 \\ \tan\lambda = \dfrac{2(0.71255\ 70540)}{1-(0.71225\ 70540)^2-(0.86229\ 25911)^2} = -5.6712\ 81819 \end{cases}$$

$$\arcsin\phi = 50°00'\ 0''\!.00000$$
$$\arctan\lambda = 100°00'\ 0''\!.00000$$

Note that $x^2+y^2>1$ since $\lambda > 90°$. For all such points add 180° to the principal value of arctan λ.

(14)
$$\mu = \ln\tan(45° + 25°) = 1.0106\ 83189$$

(17)
$$\begin{cases} x = \dfrac{0.98480\ 77530}{1.5557\ 23827+(-0.17364\ 81777)} = 0.71255\ 70540 \\ y = \dfrac{1.1917\ 53593}{1.5557\ 23827+(-0.17364\ 81777)} = 0.86229\ 25911 \end{cases}$$

as above.

Ex. 2. The Ordinary Lagrange Projection.

Using (18) with $n = 2$, $\frac{\lambda}{2} = 50°$, $\frac{\mu}{2} = \frac{1}{2}(1.0106\ 83189)$ from example 1,

$$\sin\frac{\lambda}{2} = 0.76604\ 44431$$
$$\cos\frac{\lambda}{2} = 0.64278\ 76097$$
$$\sinh\frac{\mu}{2} = 0.52712\ 60889$$
$$\cosh\frac{\mu}{2} = 1.1304\ 25545$$

(18)
$$\begin{cases} x = \dfrac{0.76604\ 44431}{1.1304\ 25545+0.64278\ 76097} = 0.43200\ 92263 \\ y = \dfrac{0.52712\ 60889}{1.1304\ 25545+0.64278\ 76097} = 0.29727\ 16999. \end{cases}$$

Using the stereographic eq (1) with $\lambda = \frac{\lambda}{2} = 50°$ and substituting $\tan\frac{\phi}{2} = \tan 25° = 0.46630\ 76582$ for $\sin\phi$ and, hence, 0.88462 26132 for $\cos\phi$, gives

$$(1)\quad \begin{cases} x = \dfrac{(0.76604\ 44431)\ (0.88462\ 26132)}{1+(0.64278\ 76097)\ (0.88462\ 26132)} = 0.43200\ 92262 \\ \\ y = \dfrac{0.46630\ 76582}{1+(0.64278\ 76097)\ (0.88462\ 26132)} = 0.29727\ 16999. \end{cases}$$

The inverse, from (22)

$$(22)\quad \begin{cases} \tan\dfrac{\lambda}{2} = \dfrac{2(0.43200\ 92263)}{1-(0.43200\ 92263)^2-(0.29727\ 16999)^2} = 1.1917\ 53593 \\ \\ \tan\dfrac{\phi}{2} = \dfrac{2(0.29727\ 16999)}{1+(0.43200\ 92263)^2+(0.29727\ 16999)^2} = 0.46630\ 76582 \end{cases}$$

$$\arctan\frac{\lambda}{2} = 50°00'\ 0.''0000 \qquad \lambda = 100°00'\ 0.''0000$$

$$\arctan\frac{\phi}{2} = 25°00'\ 0.''0000 \qquad \phi = 50°00'\ 0.''0000$$

Ex. 3. The Epicycloidal Projection.
The Lagrange coordinates from example 2 are

$$x = 0.43200\ 92263$$

$$y = 0.29727\ 16999$$

From (29)

$$(29)\quad \begin{cases} X = \dfrac{0.43200\ 92263}{2}\ (3+(0.43200\ 92263)^2-3(0.29727\ 16999)^2) = 0.63106\ 19229 \\ \\ Y = \dfrac{0.29727\ 16999}{2}\ (3+3(0.43200\ 92263)^2-(0.29727\ 16999)^2) = 0.51599\ 31360 \end{cases}$$

The Inverse

$$\left.\begin{aligned} u &= 0.63106\ 19229 \\ v &= 0.51599\ 31360 \end{aligned}\right\}\ \text{same as } X, Y \text{ above}$$

$$1+v^2+u^2 = 1.6644\ 88067$$

$$(36)\quad \sin^2 3\eta = \frac{1}{2}\ (1.6644\ 88067 - \sqrt{(1.6644\ 88067)^2-4(51599\ 31360)^2} = 0.17926\ 53088$$

$$\sin 3\eta = 0.42339\ 73416$$

$$\eta = 0.14573\ 07016 \text{ radians}$$

(37) $$\cosh 3\zeta = \frac{0.51599\ 31360}{0.42339\ 73415} = 1.2186\ 97155$$

$$\zeta = 0.21662\ 06631$$

(34) $$\begin{cases} x = 2\sinh\xi\cos\eta = 2(0.21831\ 87789)\ (0.98940\ 00608) \\ \quad = 0.43200\ 92262 \\ y = 2\cosh\xi\sin\eta = 2(1.0235\ 54145)\ (0.14521\ 54246) \\ \quad = 0.29727\ 16999 \end{cases}$$

The Lagrange coordinates are reproduced, and an inversion of these as in example 2 will produce the geographic coordinates or, directly with (39).

(39) $$\begin{cases} \tan\frac{\lambda}{2} = \dfrac{(0.21831\ 87789)\ (0.98940\ 00608)}{0.25 - ((1.0235\ 54145)^2 - (0.98940\ 00608)^2)} = 1.1917\ 53591 = \arctan 50° \\ \tan\frac{\phi}{2} = \dfrac{(1.0235\ 54145)\ (0.14521\ 54246)}{0.25 + ((1.0235\ 54145)^2 - (0.98940\ 00608)^2)} = 0.46630\ 76581 = \arctan 25° \end{cases}$$

Ex. 4. The General Lagrange Projection.

Assume a map is to be constructed to include, within a unit circle, all of the Eurasian continent, Africa and Australia. (See fig. 2.) The longitudinal extent is 220°, from 30°W to 190°E, with a central meridian 80°E. The circumference of the circle that represents the meridians ±90° in the stereographic will now represent the meridians ±110° so that $n = \frac{110}{90} = \frac{11}{9}$. For the point $\lambda = 100°$, $\phi = 50°$, $\frac{\lambda}{n} = \frac{900°}{11} = \left(81\frac{9}{11}\right)^{\circ}$, and $\frac{\mu}{n} = \frac{9}{11}(1.0106\ 83189) = 0.82692\ 26090$

$$\sin\frac{\lambda}{n} = 0.98982\ 14419$$

$$\cos\frac{\lambda}{n} = 0.14231\ 48383$$

$$\sinh\frac{\mu}{n} = 0.92443\ 94512$$

$$\cosh\frac{\mu}{n} = 1.3618\ 32699$$

(18) $$\begin{cases} x = \dfrac{0.98982\ 14419}{1.3618\ 32699 + 0.14231\ 48383} = 0.65806\ 14053 \\ y = \dfrac{0.92443\ 94512}{1.5041\ 47537} = 0.61459\ 36011 \end{cases}$$

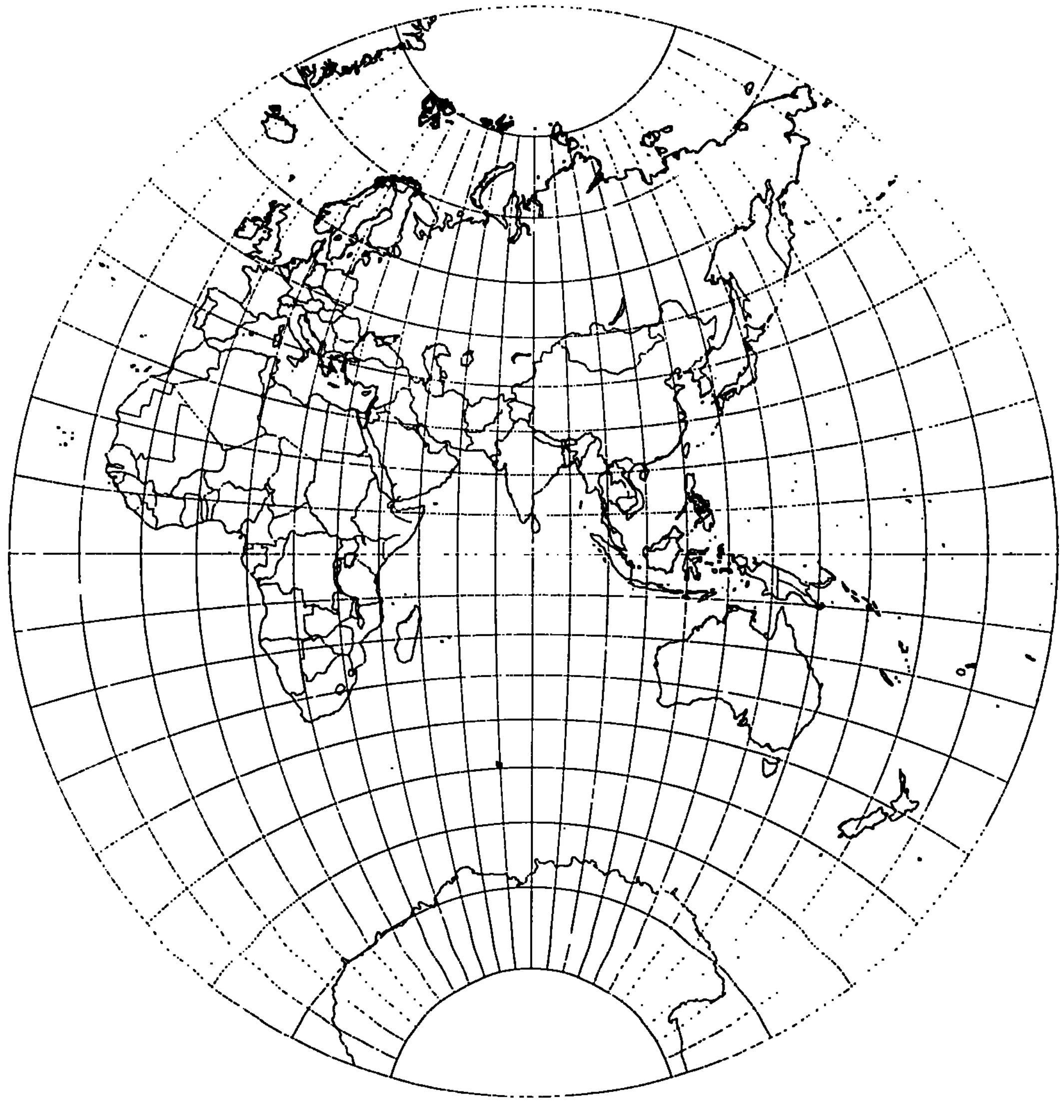

FIGURE 2.—The Eurasian Continent and Africa on a Lagrange projection.

The inverse

$$
(20)\quad \begin{cases} \tan \dfrac{9}{11}\lambda = \dfrac{2(0.65806\ 14053)}{1-(0.65806\ 1405)^2-(0.61459\ 36011)^2} = 6.9551\ 52771 \\[2ex] \tanh \dfrac{9}{11}\mu = \dfrac{2(0.61459\ 36011)}{1+(0.65806\ 14053)^2+(0.61459\ 36011)^2} = 0.67882\ 01313 \end{cases}
$$

$$\frac{9}{11}\lambda = 81.^\circ 81818182 \qquad \lambda = 100^\circ 00'\ 0.''0000$$

$$\frac{9}{11}\mu = 0.8269226089 \qquad \mu = 1.0106\ 83189$$

$$\tanh \mu = 0.76604\ 44431 = \sin \phi$$

$$\phi = 50^\circ 00' 0.''0000$$

or

$$e^{\mu}=2.7474\quad 77419=\tan\left(45^\circ+\frac{\phi}{2}\right);\ 45^\circ+\frac{\phi}{2}=70^\circ 00'0.''0000 \tag{14}$$

10. GEOMETRIC CONSTRUCTION OF POINTS ON THE EPICYCLOIDAL MAP

August shows the following geometric construction of a point U of his projection with given ϕ and λ from the defining eq (27), $u=\frac{1}{2}(3p+p^3)$, using the complex number or vector p, the image of the ordinary Lagrange projection ($n=2$), as an intermediate construction. For the sake of convenience he actually plots the value $2u=3p+p^3$ which merely represents a doubling of the scale in eq (27). The Lagrange point P in figure 3 in turn is plotted as a point of the stereographic projection with longitude $\frac{\lambda}{2}$ and latitude ϕ^* such that $\sin\phi^*=\tan\frac{\phi}{2}$. The construction of the stereographic grid itself will be assumed as known since it can be found in many texts, e.g., Deetz and Adams (1945), p. 46.

Construct a unit circle with perpendicular diameters NS and AA′. The arc NA′B is made equal to λ, chord BN intersects A′A at C. With C as center, and radius CN, draw the portion of the circular arc

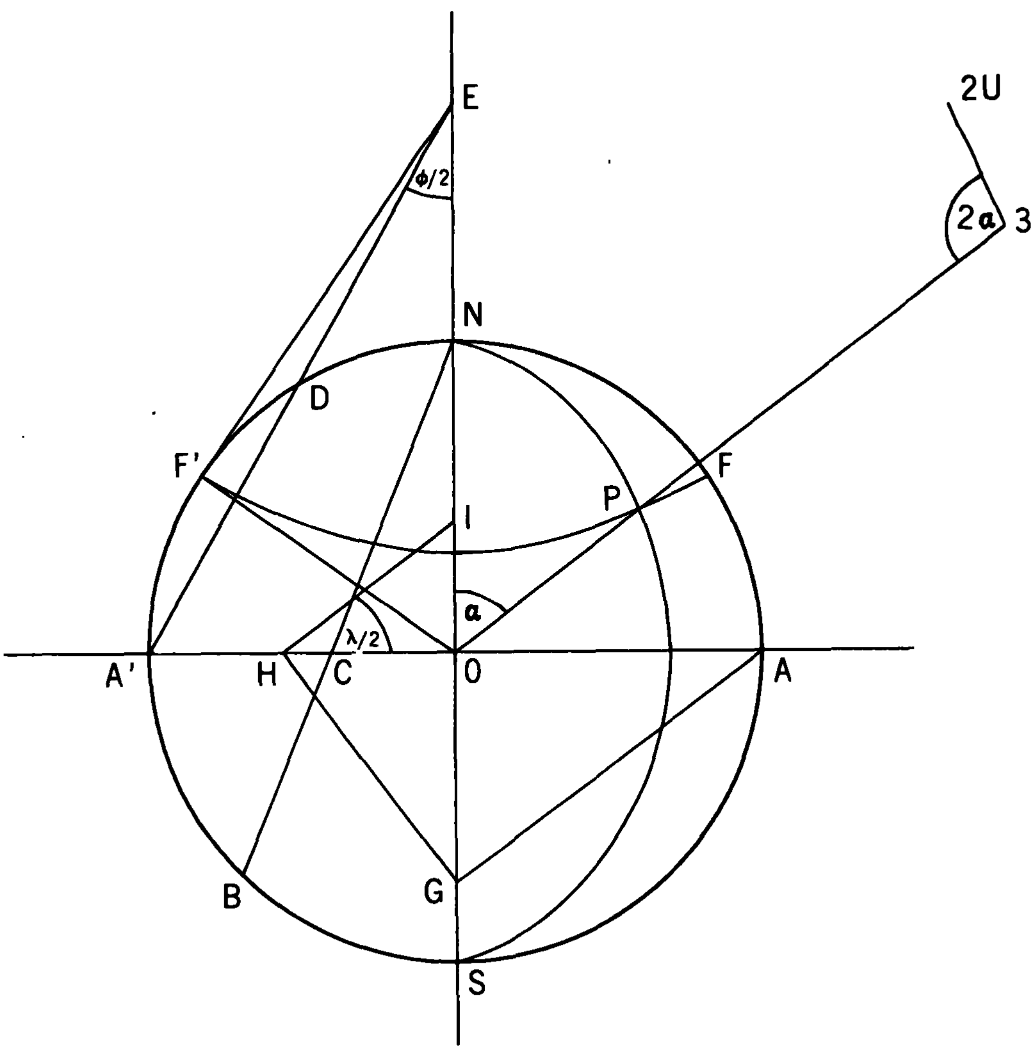

FIGURE 3. — Geometric construction of the August projection.

lying inside the unit circle. This arc is the meridian λ of the Lagrange image P. Construct arc $A'D=\phi$ and intersect chord $A'D$ with SN at E. With center E draw a circular arc cutting the unit circle orthogonally at F and F'. (This can be accomplished by bisecting EO at K and making KF and KF' = KO. OFE will then be a right angle and FE a tangent to the unit circle at F.) The circular arc FF' is the Lagrangian parallel ϕ and its intersection P with the meridian λ is the endpoint of the vector $OP=p$. Extend OP to 3P so that $O(3P)=3$ OP. At 3P construct the angle $O(3P)(2U)=2\angle$ NOP. The inclination of line (3P)(2U) will then be 3 times that of OP as called for in eq (25). The length of (3P)(2U) is OP^3 which can be constructed graphically as indicated in figure 3 by making OG = OP, drawing AG, then GH $\perp$ AG and HI $\perp$ HG. OI is then $=OP^3$, as follows from the continued proportion $\frac{OA}{OG}=\frac{OG}{OH}=\frac{OH}{OI}$ in these congruent right triangles.

To prove that the construction of the point P is in accordance with stereographic construction with the transformed longitude $\frac{\lambda}{2}$ and a latitude ϕ^* such that $\sin\phi^*=\tan\frac{\phi}{2}$, note that $\angle NCO$ is $\frac{\lambda}{2}$ because its complement in rt. $\angle NCO$, the $\angle CNO=\frac{1}{2}$ (arc SBN-arc BN) $=\frac{1}{2}(\pi-\lambda)=90°-\frac{\lambda}{2}$. For the latitude construction, $\angle OEA'=\frac{\phi}{2}$, since it is measured by half the difference between arc $SA'=90°$ and $ND=90°-\phi$. In rt $\angle EOA'$, therefore, $OE=\cot\frac{\phi}{9}$ and in rt $\angle EOF'$, $\sin\angle EOF'=\frac{OF'}{OE}=\frac{1}{\cot\frac{\phi}{2}}=\tan\frac{\phi}{2}$. But $\angle OEF'=\angle F'OA$ is the latitude angle in the construction of the stereographic projection. Hence $\sin\phi^*=\tan\frac{\phi}{2}$.

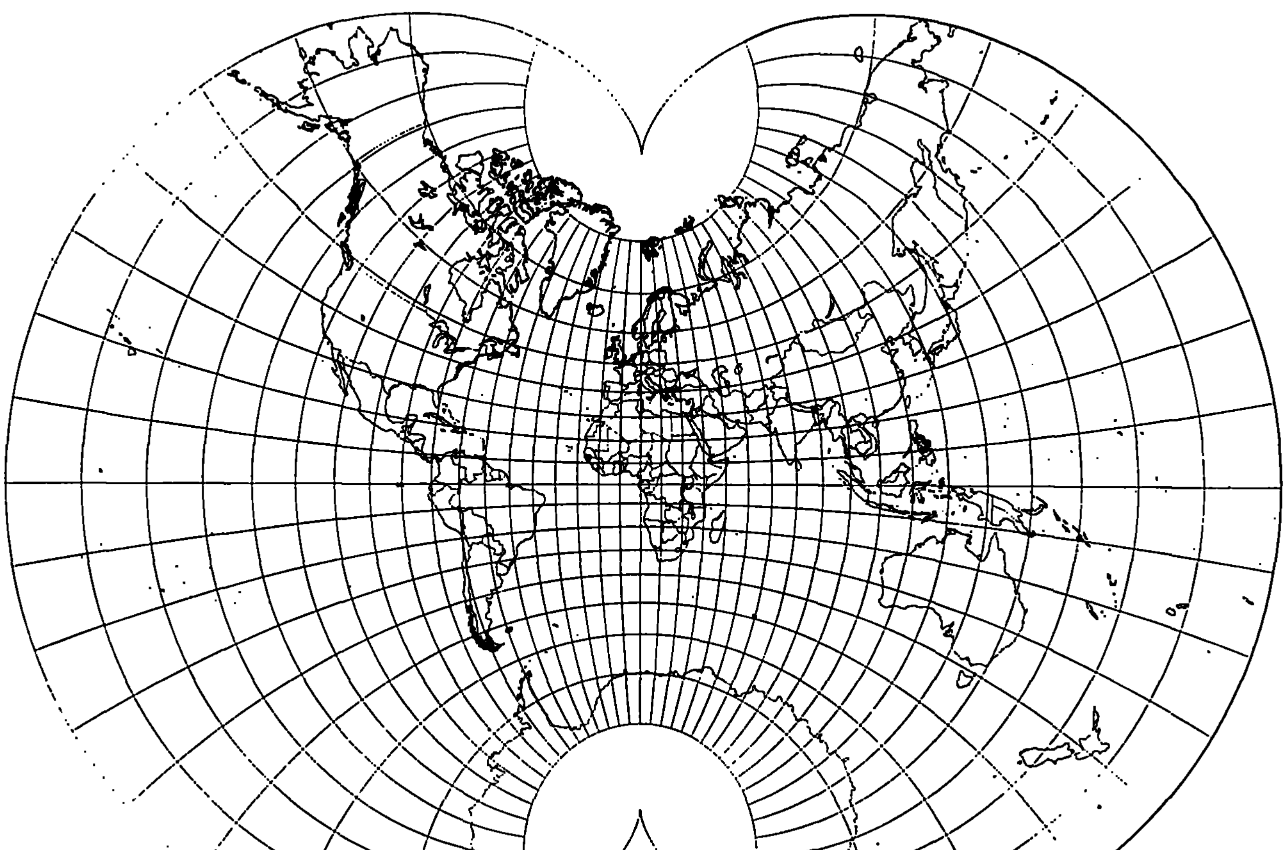

FIGURE 4.—World map on the August projection.

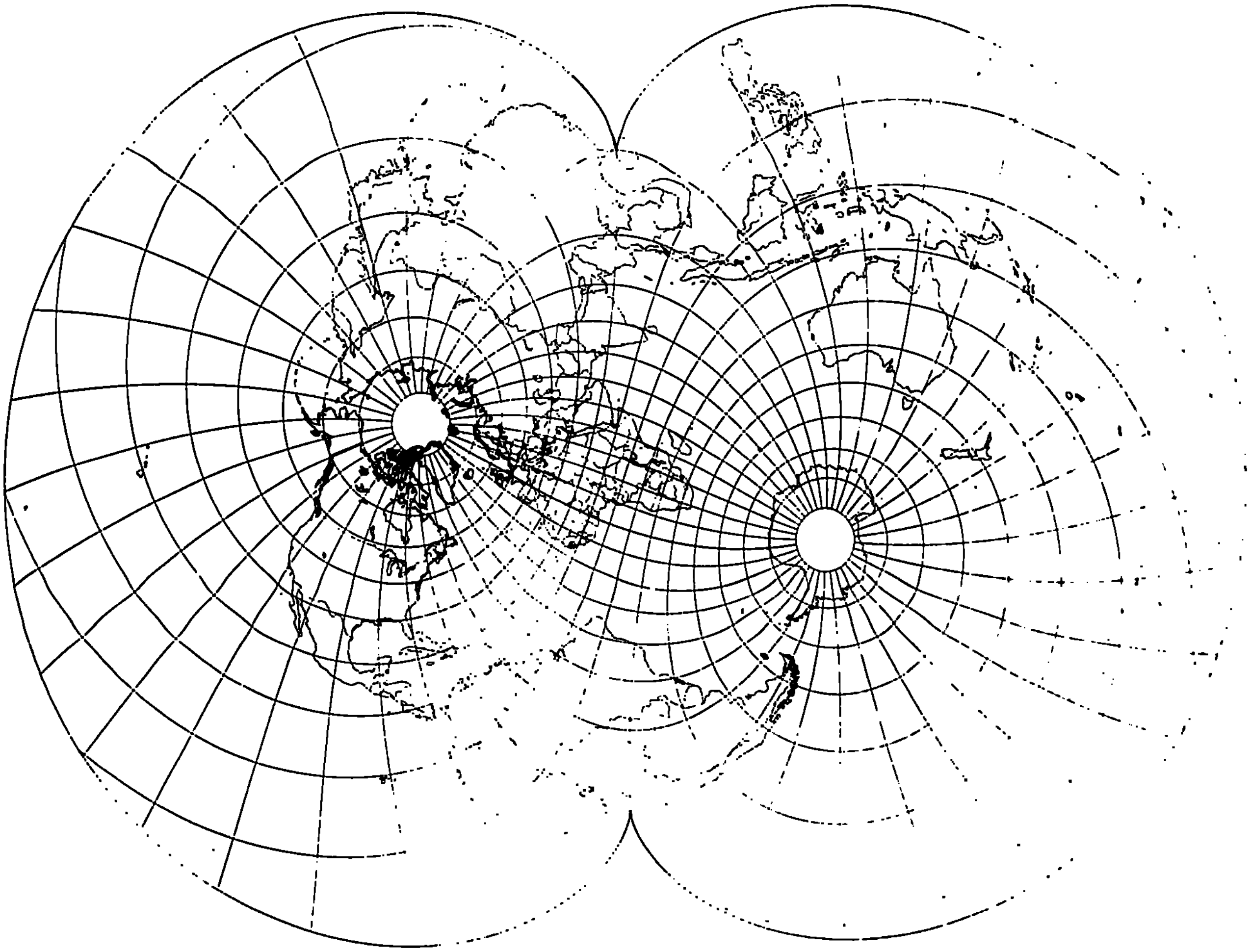

FIGURE 5. — Example of an oblique aspect of the two-cusped epicycloidal projection.

11. THE OBLIQUE PROJECTION

Figure 5 is an illustration of the epicycloidal projection in a skewed position. This is one of several such designs programmed for computation and recorded on a CDC 6600 computer equipped with an FR–80 microfilm recorder by Robert H. Hanson of the Geodetic R&D Laboratory. To incorporate the drawing of the grid with the outlines of the continents, he found it convenient to work in Cartesian space coordinates X, Y, Z on the unit sphere, instead of directly in terms of latitude and longitude, by means of the elementary transformation

$$X = \cos\phi\cos\lambda$$
$$Y = \cos\phi\sin\lambda$$
$$Z = \sin\phi$$

To effect the required skew, he then rotates the sphere, or equivalently, the coordinate axes with an orthogonal transformation matrix and uses the rotated $X'Y'Z'$ coordinates in the projection formula. This method is equivalent to the more conventional method of computing latitudes and longitudes ϕ', λ' with reference to a new set of poles. Another output of this set of programs is the appended table 1 of x, y coordinates for the ordinary Lagrange and two-cusped epicycloidal projections to replace the less precise lists in Deetz and Adams (1945), pp. 219–220.

TABLE 1. — Plane (x, y) coordinates for the Ordinary Lagrange ($n=2$) and the August two-cusped epicycloidal projections.

.... 00 DEGREES EAST LONGITUDE ....

LATITUDE (DEG)	LAGRANGE X	LAGRANGE Y	NEPHROID XX	NEPHROID YY
00	0.0000000000000	0.0000000000000	0.000000000000	0.000000000000
10	0.0000000000000	.04382836135479	-.000000000000	.0657004465290
20	0.0000000000000	.08885963229379	-.000000000000	.1329386290906
30	0.0000000000000	.13646973766161	-.000000000000	.2034338010256
40	0.0000000000000	.18844788261589	-.000000000000	.2793256863653
50	0.0000000000000	.24742760426751	-.000000000000	.3635675955548
60	0.0000000000000	.31783724519578	-.000000000000	.4607018267857
70	0.0000000000000	.40853693914747	-.000000000000	.5787125046446
80	0.0000000000000	.54346613670661	-.000000000000	.7349413646192
90	0.0000000000000	1.0000000000000	0.000000000000	1.000000000000

.... 10 DEGREES EAST LONGITUDE ....

LATITUDE (DEG)	LAGRANGE X	LAGRANGE Y	NEPHROID XX	NEPHROID YY
00	.04366094290851	0.0000000000000	.0655330293088	0.000000000000
10	.04357691392974	.04391174959618	.0652807057916	.0659503673157
20	.04331554262076	.08902768340232	.0644989751915	.1334392670288
30	.04284628082499	.13672503283094	.0631073144160	.2041860990060
40	.04210757856025	.18879433586634	.0609474175613	.2803289892834
50	.04098321880418	.24787034280421	.0577322509867	.3648154671561
60	.03924273489707	.31838181952471	.0529274426373	.4621714853410
70	.03636225630920	.40918553663951	.0454350720328	.5803343102789
80	.03074813372043	.54419573592507	.0324776924236	.7364838555933
90	0.0000000000000	1.0000000000000	0.000000000000	1.000000000000

.... 20 DEGREES EAST LONGITUDE ....

LATITUDE (DEG)	LAGRANGE X	LAGRANGE Y	NEPHROID XX	NEPHROID YY
00	.08748866352592	0.0000000000000	.1315678260509	0.000000000000
10	.08731932046244	.04416318595831	.1310564112209	.0667068054487
20	.08679260444044	.08953437626802	.1294721621623	.1349543803124
30	.08584703784233	.13749471308806	.1266525096950	.2064623599210
40	.08435878257176	.18983871036844	.1222780636190	.2833637454670
50	.08209410177441	.24920470435319	.1157703537455	.3685881293695
60	.07858974515883	.32002260248376	.1060542263605	.4666112882013
70	.07279359796932	.41113875346259	.0909262533453	.5852275748492
80	.06150935466894	.54639072974151	.0648356240918	.7411263999782
90	0.0000000000000	1.0000000000000	0.000000000000	1.000000000000

TABLE 1. — Plane (x, y) coordinates for the Ordinary Lagrange $(n=2)$ and the August two-cusped epicycloidal projections. — Continued

.... 30 DEGREES EAST LONGITUDE ....

LATITUDE	************ LAGRANGE ************		************ NEPHROID ************	
(DEG)	X	Y	XX	YY
00	.13165249758739	0.00000000000000	.1986196719471	0.000000000000000
10	.13139522827946	.04458652670083	.1978352800488	.0679901318855
20	.13059509209588	.09038741106214	.1954058727446	.1375242357875
30	.12915891063532	.13879028179580	.1910837497682	.2103216362350
40	.12689906585999	.19159620204703	.1843828236493	.2885056581922
50	.12346167184193	.25144930181450	.1744243369402	.3749739727240
60	.11814602960479	.32278095559281	.1595795969415	.4741148550309
70	.10936300863740	.41441901854888	.1365250014283	.5934765631755
80	.09229573450114	.55006977414924	.0969468965123	.7489141533948
90	0.00000000000000	1.00000000000000	0.000000000000000	1.000000000000000

.... 40 DEGREES EAST LONGITUDE ....

LATITUDE	************ LAGRANGE ************		************ NEPHROID ************	
(DEG)	X	Y	XX	YY
00	.17632698070846	0.00000000000000	.2672315801377	0.000000000000000
10	.17597775968288	.04518833906126	.2661524781018	.0698354719924
20	.17489176138940	.09159989774040	.2628112056361	.1412182269728
30	.17294292764446	.14063131465897	.2568702069793	.2158655945490
40	.16987758351234	.19409265075386	.2476681407827	.2958848542282
50	.16521769091811	.25463574851896	.2340125854580	.3841245569978
60	.15801802757611	.32669309404503	.2137023609068	.4848420704761
70	.14613923368740	.41906423740542	.1822730485035	.6052241361149
80	.12311726694374	.55526418652250	.1286700348749	.7599221464761
90	0.00000000000000	1.00000000000000	0.000000000000000	1.000000000000000

.... 50 DEGREES EAST LONGITUDE ....

LATITUDE	************ LAGRANGE ************		************ NEPHROID ************	
(DEG)	X	Y	XX	YY
00	.22169466264294	0.00000000000000	.3379899766266	0.000000000000000
10	.22124791560833	.04597811977664	.3365854139728	.0722945686745
20	.21985883152440	.09319078668116	.3322379515032	.1461384924715
30	.21736685869587	.14304608773133	.3245137001584	.2232436516684
40	.21344915531340	.19736533789643	.3125644106160	.3056921209032
50	.20749808285368	.25880957530130	.2948660081857	.3962612659834
60	.19831432136098	.33181103169363	.2686200093439	.4990250691330
70	.18319057086314	.42512859321697	.2281964477139	.6206754248049
80	.15398070993185	.56201827801347	.1598409404780	.7742548230596
90	0.00000000000000	1.00000000000000	0.000000000000000	1.000000000000000

TABLE 1.—Plane (x, y) coordinates for the Ordinary Lagrange ($n=2$) and the August two-cusped epicycloidal projections.—Continued

.... 60 DEGREES EAST LONGITUDE ....

LATITUDE (DEG)	LAGRANGE X	LAGRANGE Y	NEPHROID XX	NEPHROID YY
00	.26794919243112	0.00000000000000	.4115427318801	0.000000000000
10	.26739760359025	.04696861838767	.4097712282032	.0754386084752
20	.26568283823788	.09518550526013	.4042904423080	.1524253959997
30	.26260776559497	.14607250427078	.3945617744942	.2326607307963
40	.25777636884098	.20146416063600	.3795351332772	.3181882807768
50	.25044442000107	.26403157512379	.3573321436791	.4116852333766
60	.23914631173810	.33820395745152	.3245269506811	.5169770404110
70	.22058454411615	.43268369896480	.2742982031290	.6401030567305
80	.18488827960126	.57038981063770	.1902639125888	.7920451424577
90	0.00000000000000	1.00000000000000	0.000000000000	1.000000000000

.... 70 DEGREES EAST LONGITUDE ....

LATITUDE (DEG)	LAGRANGE X	LAGRANGE Y	NEPHROID XX	NEPHROID YY
00	.31529878887898	0.00000000000000	.4886206340039	0.000000000000
10	.31463303939372	.04817628453035	.4864275692920	.0793622603312
20	.31256382529492	.09761683752169	.4796462252655	.1602653416201
30	.30885483131342	.14975937282353	.4676228460675	.2443882808062
40	.30303187604739	.20645324552155	.4490871024130	.3337174272591
50	.29420549324058	.27037964098437	.4217790883226	.4307912134027
60	.28062884774208	.34596009992563	.3816113675483	.5391043062753
70	.25838740660353	.44182013160616	.3205488233368	.6638539949655
80	.21583613753149	.58045056855727	.2197014012992	.8134529613312
90	0.00000000000000	1.00000000000000	0.000000000000	1.000000000000

.... 80 DEGREES EAST LONGITUDE ....

LATITUDE (DEG)	LAGRANGE X	LAGRANGE Y	NEPHROID XX	NEPHROID YY
00	.36397023426620	0.00000000000000	.5700637081220	0.000000000000
10	.36317865538257	.04962186678334	.5673779874534	.0841892997173
20	.36071899287865	.10052610017506	.5590786739216	.1699016287487
30	.35631256197661	.15416811127304	.5443841703735	.2587795090177
40	.34940094357990	.21241309067744	.5217819270068	.3527250787526
50	.33893898327802	.27795118929555	.4885990239406	.4540863618846
60	.32288078150779	.35518915840694	.4400500171350	.5659222927377
70	.29666340420305	.45264938871739	.3668739560154	.6923580331189
80	.24681260454070	.59228702663657	.2478621376253	.8386622790448
90	0.00000000000000	1.00000000000000	0.000000000000	1.000000000000

TABLE 1.—Plane (x, y) coordinates for the Ordinary Lagrange ($n=2$) and the August two-cusped epicycloidal projections.—Continued

.... 90 DEGREES EAST LONGITUDE ....

LATITUDE (DEG)	*********** LAGRANGE *********** X	Y	*********** NEPHROID *********** XX	YY
00	.41421356237309	0.00000000000000	.6568542494924	0.000000000000
10	.41328168025204	.05133120166092	.6535837078679	.0900803647951
20	.41038694810823	.10396468922638	.6434849858435	.1816493758612
30	.40520447463426	.15937498068339	.6256335841173	.2762902045294
40	.39708432531229	.21944335935885	.5982491897340	.3757827629774
50	.38481324709252	.28686629407625	.5582109472675	.4822152744736
60	.36602540378444	.36602540378443	.5000000000000	.5980762113533
70	.33547371034843	.46530631735255	.4131380051275	.7261379292038
80	.27779602406123	.60600108984498	.2743872818761	.8678767561416
90	0.00000000000000	1.00000000000000	0.000000000000	1.000000000000

.... 100 DEGREES EAST LONGITUDE ....

LATITUDE (DEG)	*********** LAGRANGE *********** X	Y	*********** NEPHROID *********** XX	YY
00	.46630765815499	0.00000000000000	.7501591161313	0.000000000000
10	.46521759859264	.05333624627916	.7461841735525	.0972436439229
20	.46183273963494	.10799609999772	.7339214997044	.1959160101470
30	.45577740897909	.16547399166545	.7122862276131	.2975071373216
40	.44630153555286	.22766649206817	.6792015295920	.4036211757368
50	.43200922628604	.29727169983052	.6310619228960	.5159931359518
60	.41019069327069	.37863158384175	.5615860844502	.6363674922104
70	.37487491469343	.47995207366969	.4591222829345	.7658210346732
80	.30875218572006	.62171086325342	.2988342503665	.9013126843318
90	0.00000000000000	1.00000000000000	0.000000000000	1.000000000000

.... 110 DEGREES EAST LONGITUDE ....

LATITUDE (DEG)	*********** LAGRANGE *********** X	Y	*********** NEPHROID *********** XX	YY
00	.52056705055174	0.00000000000000	.8513848224287	0.000000000000
10	.51929675937561	.05567642891394	.8465496656709	.1059496694155
20	.51535390844020	.11269856247549	.8316489562968	.2132295112137
30	.50830663455411	.17258067908855	.8054178000692	.3231869030780
40	.49729461294838	.23723236401769	.7654518683202	.4371749557997
50	.48072247163883	.30934593646078	.7076257303228	.5564495178849
60	.45550928010592	.39320380520550	.6248815250717	.6817872606328
70	.41491690961946	.49677767906717	.5044957805804	.8121520365556
80	.33963120546789	.63955139255002	.3206579466957	.9391882817167
90	0.00000000000000	1.00000000000000	0.000000000000	1.000000000000

TABLE 1. — Plane (x, y) coordinates for the Ordinary Lagrange $(n=2)$ and the August two-cusped epicycloidal projections. — Continued

.... 120 DEGREES EAST LONGITUDE ....

LATITUDE (DEG)	*********** LAGRANGE *********** X	Y	*********** NEPHROID *********** XX	YY
00	.57735026918961	0.00000000000000	.9622504486493	0.000000000000
10	.57587248647246	.05840042085852	.9563506589009	.1165519611074
20	.57128785553082	.11816848891309	.9381913218514	.2342776348821
30	.56310196894106	.18083701467352	.9063063544441	.3543093927729
40	.55033247757727	.24832429598726	.8579328272773	.4776434087997
50	.53116525700269	.32330583280214	.7883970502553	.6048862883392
60	.50211797591007	.40997761055292	.6898793537819	.7355584076616
70	.45563996661702	.51600824224177	.5487760247330	.8660060113052
80	.37036374307606	.65967528700761	.3391892730087	.9817077917952
90	0.00000000000000	1.00000000000000	0.000000000000	1.000000000000

.... 130 DEGREES EAST LONGITUDE ....

LATITUDE (DEG)	*********** LAGRANGE *********** X	Y	*********** NEPHROID *********** XX	YY
00	.63707026080748	0.00000000000000	1.0848855869147	0.000000000000
10	.63535116151572	.06156847485269	1.0776505762620	.1295161399849
20	.63002092132716	.12452500899256	1.0554132297103	.2599629294083
30	.62051518009228	.19041793088276	1.0164852262499	.3921520331094
40	.60571599565030	.26116683889024	.9577180079317	.5265733929173
50	.58356872030097	.33941482291462	.8738781645701	.6629543544720
60	.55015665265823	.42923552613566	.7564496991728	.7991880225485
70	.49707072478378	.53790791343895	.5912766957561	.9284002786318
80	.40085642234277	.68225309786651	.3536110840878	1.0290383646726
90	0.00000000000000	1.00000000000000	0.000000000000	1.000000000000

.... 140 DEGREES EAST LONGITUDE ....

LATITUDE (DEG)	*********** LAGRANGE *********** X	Y	*********** NEPHROID *********** XX	YY
00	.70020753820970	0.00000000000000	1.2219638931289	0.000000000000
10	.69820491438156	.06525553636938	1.2130316309007	.1454615083842
20	.69199970153061	.13191598131023	1.1856231941797	.2914808224535
30	.68094902139970	.20153927798420	1.1378104291009	.4383939439214
40	.66378387456850	.27603590926753	1.0660440589841	.5859733444571
50	.63818491103281	.35799357002951	.9645532463878	.7327556565805
60	.59976614997079	.45131642240182	.8242765814802	.8745320318767
70	.53921672058434	.56278562172427	.6310376991805	1.0005023702529
80	.43098630627472	.70747327163858	.3629322552873	1.0812770891406
90	0.00000000000000	1.00000000000000	0.000000000000	1.000000000000

TABLE 1. — Plane (x, y) coordinates for the Ordinary Lagrange ($n = 2$) and the August two-cusped epicycloidal projections. — Continued

.... 150 DEGREES EAST LONGITUDE ....

LATITUDE (DEG)	LAGRANGE X	LAGRANGE Y	NEPHROID XX	NEPHROID YY
00	.76732698797895	0.00000000000000	1.3768879816454	0.000000000000
10	.76498779144456	.06955542442012	1.3657680655733	.1652212946036
20	.75774530532219	.14052603674422	1.3317128408601	.3304319585255
30	.74486834633001	.21447005129460	1.2725467305929	.4952638032231
40	.72491950985350	.29327207461349	1.1843310027809	.6584715270651
50	.69528852800531	.37943360901861	1.0608422238016	.8169790969339
60	.65108473962598	.47662710943896	.8927645086570	.9638736169790
70	.58205896484134	.59100160938421	.6667322266881	1.0836295273308
80	.46059427288101	.73554142566919	.3659613836645	1.1384048433704
90	0.00000000000000	1.00000000000000	0.000000000000	1.000000000000

.... 160 DEGREES EAST LONGITUDE ....

LATITUDE (DEG)	LAGRANGE X	LAGRANGE Y	NEPHROID XX	NEPHROID YY
00	.83909963117727	0.00000000000000	1.5540495174756	0.000000000000
10	.83635661052379	.07458651469730	1.5400682901570	.1899313521480
20	.82787151635138	.15058745840367	1.4973469962589	.3789862544892
30	.81281387640295	.22954644174686	1.4234775648756	.5657525126940
40	.78955888326227	.31329810389986	1.3141955444229	.7475377331536
50	.75517797474199	.40421494541515	1.1630211314259	.9190820341336
60	.70424245434803	.50565667618457	.9609005928167	1.0700159535095
70	.62554191188146	.62297470512240	.6965441197321	1.1792321109185
80	.48947714364351	.76667859766766	.3612829959699	1.2002229054137
90	0.00000000000000	1.00000000000000	0.000000000000	1.000000000000

.... 170 DEGREES EAST LONGITUDE ....

LATITUDE (DEG)	LAGRANGE X	LAGRANGE Y	NEPHROID XX	NEPHROID YY
00	.91633117401742	0.00000000000000	1.7592013700590	0.000000000000
10	.91309820726558	.08049956673918	1.7414188049585	.2211630939974
20	.90310817868611	.16239508134503	1.6872263888017	.4401264773250
30	.88541935800008	.24719273431048	1.5940446845825	.6539234208865
40	.85819954677908	.33664235493419	1.4574469220740	.8577969220886
50	.81817510492653	.43292886801185	1.2710875065467	1.0435326314376
60	.75935226140995	.53899416889173	1.0270511618828	1.1963872617454
70	.66955995893370	.65919014667811	.7180078412409	1.2888485636923
80	.51737844660932	.80111799712787	.3472412143861	1.2662676652415
90	0.00000000000000	1.00000000000000	0.000000000000	1.000000000000

TABLE 1.—Plane (x, y) coordinates for the Ordinary Lagrange ($n=2$) and the August two-cusped epicycloidal projections.—Continued

.... 180 DEGREES EAST LONGITUDE ....

LATITUDE (DEG)	*********** LAGRANGE *********** X	Y	*********** NEPHROID *********** XX	YY
00	.99999999999999	0.00000000000000	2.0000000000000	0.0000000000000
10	.99616551524052	.08748866352592	1.9770811983244	.2611266675296
20	.98433164933075	.17632698070846	1.9074551861879	.5180165058252
30	.96343304400227	.26794919243112	1.7885233271941	.7653718043597
40	.93141058001731	.36397023426619	1.6160451722820	.9954772759076
50	.88462261329110	.46630765815499	1.3845355339162	1.1961324588696
60	.81649658092772	.57735026918961	1.0886621079037	1.3471506281091
70	.71393935557181	.70020753820969	.7278032060336	1.4140122713718
80	.54397776513213	.83909963117727	.3219388890265	1.3356986106928
90	0.00000000000000	1.00000000000000	0.0000000000000	1.0000000000000

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(Continued from inside front cover)

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